Optimal Confidence Bands in Simple Linear Regression

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We consider the problem of constructing optimal confidence bands for a simple linear regression over the whole real line. Optimality is defined as minimization of the average width of the band, weighted by a normalized function. We present this weight function as an indicator of experimental interest in the varying width of the band. A comparison between the commonly-seen hyperbolic bands and segmented-line bands helps motivate the discussion.

KEY WORDS: Simultaneous confidence intervals; Prior weight functions; Constrained minimization.

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1. INTRODUCTION

In both the physical and social sciences, experimental situations sometimes involve prediction of one variable from another. A common model is the simple linear regression, $Y = \beta_0 + \beta_1 x + \epsilon$. Often experimenters express additional interest by soliciting the construction of a confidence region around the regression line. The classical region, first developed by Working and Hotelling (1929) and reformulated by Scheffé (see Miller, 1981, Ch. 3), consists of hyperbolic bands which extend over all values of $x \in \mathbb{R}$. An alternative to these hyperbolic bands was proposed by Graybill and Bowden (1967). They constructed bands made up of segments of straight lines which, over certain regions in $\mathbb{R}$, were shorter than the hyperbolic bands.

Recent work in the topic has included discussions on the problem of calibration and the use of discrimination intervals (Trout and Swallow, 1979; Hunter, 1981), sharpening the bands' confidence coefficient (Khorasani and Milliken, 1979; Zvára, 1979), and constructing confidence regions over restricted sets of the predictor variable (Casella and Strawderman, 1980; Uusipaika, 1983). A number of particularly interesting works have attempted to establish an optimality standard for confidence band construction. Bowden (1970) has shown that the choice of band form can be reduced to a choice of mathematical functions in a generalized expression for a confidence region. Bohrer (1973) noted that the hyperbolic bands are optimal in the sense that they minimize average width over certain sets when no intercept is included in the model; Naiman (1983a) has worked on extending Bohrer's results.

In much of this work, however, the choice of band form is left prespecified. To date there have been few substantive attempts at unifying the theory by taking prior experimental interest into account; for example, an experimenter interested in making more precise confidence statements near $\bar{x}$ might consider the hyperbolic bands, since they attain their minimum width at $x = \bar{x}$. Cima and Hochberg (1976) have made some headway in this direction. They consider a uniform importance criterion for constructing simultaneous interval estimators over the set of normalized linear combinations of the parameter vector. Along a slightly different line, an early work by Hoel (1951) introduced the notion that prior interest can be consolidated into mathematical form. From this information, optimal confidence bands can be constructed. Naiman (1983b) has indicated progress in developing this concept when the range of the predictor variable is initially restricted.

Hoel's suggestion involved specifying a weight function and minimizing the weighted area of the resulting band. The weight function, $\tau(x)$, can be taken to represent prior experimental interest. If we denote the length of the band at t as $\ell(t)$, then this optimality criterion amounts to minimizing

$$\int_{-\infty}^{\infty} c\tau(x)k\ell(x)dx \tag{1.1}$$

subject to a $1-\alpha$ coverage probability constraint (developed in Section 2.2). Here, c is a normalizing constant that makes

$$\int_{-\infty}^{\infty} c\tau(x)dx = 1 \quad , \tag{1.2}$$

and the constant k allows the shape function, ℓ, to satisfy the $1-\alpha$ constraint. We say that the particular function $k\ell(t)$, which minimizes (1.1) subject to the $1-\alpha$ constraint is a τ-optimal band, after Naiman (1983b). In particular, Hoel (1951) considered this question of which τ makes the hyperbolic band τ-optimal. In Section 3 we will extend Hoel's work to the segmented-line bands of Graybill and Bowden (1967).

In Section 4 we will consider a general method for constructing minimal $1-\alpha$ confidence bands which are, at least approximately, τ-optimal.

2. GENERAL THEORY

2.1. Reduction to Standardized Form

Let $\hat{\beta}_0$, $\hat{\beta}_1$, and s^2 be the usual least squares estimators of β_0, β_1, and σ^2. Also, suppose that the original predictor variables were centered so that their mean is zero. Then, under the assumption $\epsilon \sim N(0,\sigma^2)$, $\hat{\beta}_0$ and $\hat{\beta}_1$ are statistically independent. Denote $u_0 = \beta_0 - \hat{\beta}_0$ and $u_1 = \beta_1 - \hat{\beta}_1$. Then, the vector $\underset{\sim}{u}$ has a bivariate Normal distribution:

$$\underset{\sim}{u} \sim N_2\left(\underset{\sim}{0}, \sigma^2 \begin{bmatrix} 1/n & 0 \\ 0 & 1/\Sigma_1^n x_i^2 \end{bmatrix}\right) .$$

The $1-\alpha$ coverage probability constraint is then of the form

$$P[\,|u_0 + u_1 t| \le k' s \ell(t) \qquad \forall t \in \mathbf{R}] \ge 1 - \alpha \quad . \tag{2.1}$$

A confidence band as denoted by (2.1) is the set

$$\{\underset{\sim}{u} : |u_0 + u_1 t| \le k' s \ell(t) \qquad \forall t \in \mathbf{R}\} . \tag{2.2}$$

Write $\underset{\sim}{v} = [n^{-\frac{1}{2}} \;\; t(nm_2)^{-\frac{1}{2}}]$ and $\underset{\sim}{w}' = \sigma^{-1}[u_0\sqrt{n} \;\; u_1\sqrt{nm_2}]$, where $m_2 = \Sigma x_i^2/n$ is the second (sample) moment of the centered x_is. Then $|u_0 + u_1 t| = \sigma|\underset{\sim}{w}'\underset{\sim}{v}|$, and $a = v_1/v_0$ gives $t = a\sqrt{m_2}$. With this, (2.2) becomes

$$\{\underset{\sim}{w} : |w_0 + w_1 a| \le k\ell^*(a) \qquad \forall a \in \mathbf{R}\} \quad , \tag{2.2'}$$

where $k = k's\sqrt{n}/\sigma$ is a constant, independent of $\underset{\sim}{u}$, with respect to the minimization. Also, from above, we have ℓ^* related to ℓ through $t = a\sqrt{m_2}$. Since $\underset{\sim}{w} \sim N_2(\underset{\sim}{0}, \underset{\sim}{I}_2)$, we see that the general setting can be reduced to this simpler, canonical case. The procedure could thus entail (i) specifying $\tau(x)$, (ii) calculating $\tau^*(a) = \tau(a\sqrt{m_2})$, (iii) solving for $\ell^*(a)$, and (iv) reporting $\ell(t) = \ell^*(t/\sqrt{m_2})$.

We could scale t to $t/\sqrt{m_2}$, so that $\ell(t) = \ell^*(t)$. Note that, in terms of the original x_i's, this is equivalent to standardizing the original scale, i.e., using $(x-\bar{x})/(\Sigma_1^n(x_i-\bar{x})^2/n)^{\frac{1}{2}}$. As such, without loss of generality, we will consider the problem from this perspective, i.e., minimizing (1.1) subject to

$$P[\,|w_0 + w_1 t| \leq k\ell(t) \quad \forall t\epsilon \mathbf{R}] \geq 1-\alpha \quad , \tag{2.3}$$

where $w_j \sim$ i.i.d. $N(0\,,1)$ $(j = 0,1)$.

2.2. Geometry of the Probability Constraint

When manipulation of the $1-\alpha$ constraint in (2.3) is involved, most authors make use of the relation

$$P[\,|w_0 + w_1 t| \leq k\ell(t) \quad \forall t\epsilon \mathbf{R}] = P[\sup_t |w_0 + w_1 t| \leq k\ell(t)] \quad ,$$

and then examine the distribution of $\sup_t|w_0 + w_1 t|$ (cf. Casella and Strawderman, 1980). Instead of using this construction, we will take a geometric approach. Consider the geometry of the solution set of $\{\underset{\sim}{w}: |w_0 + w_1 t| \leq k\ell(t)\}$. For each value of $t\epsilon\mathbf{R}$, equality is attained at a pair of parallel lines equidistant from the origin. Inequality is attained in the infinite strip between these lines. The infinite intersection of these infinite strips over all t on the real line produces a convex set: $C_\ell = \{\underset{\sim}{w}: |w_0 + w_1 t| \leq k\ell(t) \quad \forall t\epsilon\mathbf{R}\}$ (as in Wynn and Bloomfield, 1971). It is the probability content of a standard bivariate Normal density over C_ℓ to which (2.3) corresponds.

Consider the polar transform $\underset{\sim}{w}' = [r\cos\theta \quad r\sin\theta]$. As Figure 1 illustrates, the ray at angle θ from the origin can be written as $w_1 = w_0\tan\theta$. Since the statement $w_0 + w_1 t = k\ell(t)$ specifies an infinite strip for a given t, the ray at angle θ from the origin intersects the line $w_0 + w_1 t = k\ell(t)$ at a right angle. The two lines' slopes are negative reciprocals of each other, so we get $t = \tan\theta$. This point of intersection is, therefore, $w_0 = k\ell(\tan\theta)/\sec^2\theta$, $w_1 = k\ell(\tan\theta)\tan\theta/\sec^2\theta$. Thus the maximum length of a line segment in C_ℓ at angle θ is

$$\frac{(k^2\ell^2(\tan\theta) + k^2\ell^2(\tan\theta)\tan^2\theta)^{\frac{1}{2}}}{\sec^2\theta} = k\ell(\tan\theta)|\cos\theta| \quad . \qquad (2.4)$$

We will return to this relationship below.

We can restrict our attention in w-space by noting a few of the geometric characteristics of C_ℓ. First, the relationship $t = \tan\theta$ shows that considering only $t > 0$ restricts our attention to the Ist and IIIrd quadrants. Similarly, $t < 0$ puts us in the IInd and IVth quadrants. Also, since the infinite strips involved in the intersecting process which builds C_ℓ are symmetric about the origin, we see that C_ℓ is a balanced set – i.e., any line through the origin intersects C_ℓ at two points equidistant from the origin. Thus, in describing C_ℓ, we can restrict our attention to only, say, the Ist and IVth quadrants ($t > 0$ and $t < 0$, respectively). Further, if we restrict ourselves to symmetric band forms, i.e.,

$$\ell(t) = \ell(-t) \qquad (2.5)$$

(or, in general, bands symmetric about $\bar{x}$), then the distances along rays out from the origin will be the same in the Ist and IVth quadrants for any given value of $|t|$ (see Figure 1). The intersecting process in the IVth quadrant will, therefore, mimic that in the Ist quadrant. As such, we need only consider the Ist quadrant relationships in the construction of C_ℓ when ℓ is a symmetric band form.

In order to describe the probability content of C_ℓ we integrate a standard bivariate Normal over the I^{st} quadrant representation of C_ℓ, multiply the result by four, and set it greater than or equal to $1-\alpha$. Hence any shape function, ℓ, can be specified, and the value of k found using equality in (2.3) and the geometrical approach outlined above.

2.3. Incorporation of the Weight Function

We now formally define a band which minimizes (1.1) subject to (2.3) as τ-optimal. This is the problem Hoel (1951) and Naiman (1983b) have investigated. Hoel's work involved the classical hyperbolic bands, while Naiman considered bands over finite intervals (such as the uniform width bands first proposed by Gafarian [1964]). For now, we remain concerned with bands over the whole real line.

Notice that equality in the constraint (2.3) leads to a useful lower bound for $\ell(t)$. Certainly, from the fact that $w_0 + w_1 t \sim N(0\,,\,1+t^2)$,

$$\begin{aligned} P[\,|w_0 + w_1 t| \le k\ell(t) \quad \forall t] &= P\left[\frac{|w_0+w_1 t|}{(1+t^2)^{\frac{1}{2}}} \le \frac{k\ell(t)}{(1+t^2)^{\frac{1}{2}}} \qquad \forall t\right] \\ &\le P\left[\frac{|w_0+w_1 t_0|}{(1+t_0^2)^{\frac{1}{2}}} \le \frac{k\ell(t_0)}{(1+t_0^2)^{\frac{1}{2}}} \quad \text{for some } t_0\right] \\ &= \Phi\left(k\ell(t_0)/(1+t_0^2)^{\frac{1}{2}}\right) , \qquad (2.6) \end{aligned}$$

where Φ is the standard Normal c.d.f. This is true for any t_0 on the real line. Since a necessary condition for (2.3) to hold is

$$P\left[\frac{|w_0+w_1 t|}{(1+t^2)^{\frac{1}{2}}} \le \frac{k\ell(t)}{(1+t^2)^{\frac{1}{2}}}\right] \ge 1 - \alpha \qquad \forall t \ ,$$

we get, using (2.6), $k\ell(t) \ge \Phi^{-1}(1 - \frac{\alpha}{2})$, for all t. For notation's sake we take

$\Phi^{-1}(1-\frac{\alpha}{2}) = z_{\alpha/2}$, giving

$$k\ell(t) \geq z_{\alpha/2} \qquad \forall t \quad . \tag{2.7}$$

This lower bound is useful for constructing a bound on (1.1):

$$\int_{-\infty}^{\infty} c\tau(t)k\ell(t)dt \geq z_{\alpha/2} \quad , \tag{2.8}$$

using (1.2) and (2.7).

Now, given ℓ, the constraint (2.3) can be expressed as an integral expression, using the joint distribution of the random vector $[r \quad \theta]'$ (Feller, 1968, p. 68):

$$f(r,\theta) = \frac{1}{2\pi} I_{(-\pi,\pi)}(\theta) r e^{-r^2/2} I_{(0,\infty)}(r) \quad .$$

However, by only considering the I^{st} quadrant for symmetric bands, we take only $0 \leq \theta < \pi/2$. The variable r is positive, and bounded above by a function of θ, and, implicitly $k\ell(t)$; call this $R_\ell(\theta)$. Then, equality in (2.3) gives

$$4\int_0^{\pi/2} \frac{1}{2\pi} \int_0^{R_\ell(\theta)} r e^{-r^2/2} dr d\theta = \frac{2}{\pi}\int_0^{\pi/2} (1 - \exp[-R_\ell^2(\theta)/2])d\theta = 1-\alpha . \tag{2.3'}$$

Using the relationship $x = \tan\theta$ reduces this to

$$\int_0^{\infty} \frac{\exp[-R_\ell^2(\arctan x)/2]}{x^2+1} dx = \pi\alpha/2 \quad . \tag{2.9}$$

By considering symmetric band forms we can infer the symmetry of τ. Our concern then becomes one of minimizing the integral of $c\tau(x)k\ell(x)$ over $0 < x < \infty$, subject to (2.9). This is equivalent to minimizing their difference

$$\int_0^{\infty} \left(c\tau(x)k\ell(x) - \frac{\exp[-R_\ell^2(\arctan x)/2]}{x^2+1} \right) dx \quad . \tag{2.10}$$

From (2.8) and (2.9), this is bounded below by $z_{\alpha/2} - \frac{1}{2}\pi\alpha$. This difference is non-negative whenever $2\, z_{\alpha/2} \geq \alpha\pi$, or $\alpha \leq 0.465068$. Thus, for any α in this range, (2.10) attains a minimum at zero. Setting the integrand in (2.10) equal to zero yields

$$c\tau(x) = \frac{\exp[-R_{\ell}^2(\arctan x)/2]}{k\ell(x)(x^2+1)} . \tag{2.11}$$

Coupled with the provision that $\tau(x) = \tau(-x)$ and $\ell(x) = \ell(-x)$, this gives a relationship between τ and ℓ. When $R_{\ell}(\arctan x)$ is directly proportional to $k\ell(x)$ — as it is, for example, with the currently-available band forms over the whole real line — (2.11) inversely relates the weight and band form functions. This is intuitively appealing, since a low value for $\tau(x)$ indicates little prior interest in the band at that x. The result is a large value for the width at x.

3. A COMPARISON OF EXISTING BAND FORMS

It is of some pedagogic interest to determine the form of the weight function against which the currently-available band forms are τ-optimal. The formulation in (2.11) provides an easy solution to this question.

3.1. Hyperbolic Bands

In our canonical setting, the bands of Working and Hotelling (1929) involve the hyperbola $\ell_{WH}(x) = (1+x^2)^{\frac{1}{2}}$. The constant k satisfies $P[\chi^2(2) \leq k^2] = 1 - \alpha$. As Hoel points out (1951, sec. 4), the solution set, C_{WH}, of w-vectors for this band form function is a circle of radius k, centered at the origin. A circle has the property that any ray out from its center intersects the tangent to its boundary at a right angle. Thus the boundary of C_{WH} is described by the intersections of $w_1 = w_0 \tan\theta$ and $w_0 + w_1 \tan\theta = k\ell(\tan\theta)$ (see Figure 2). This gives $R_{\ell}(\theta) = k\ell_{WH}(\tan\theta)|\cos\theta|$, or simply $k\ell_{WN}(\tan\theta)\cos\theta$ for $0 \leq \theta < \pi/2$. Substituting this into (2.11) gives, whenever $\alpha < 0.4651$, $\tau(x) = (1+x^2)^{-3/2}$. To satisfy (1.2) we get

$$c\tau_{WH}(x) = \tfrac{1}{2}(1+x^2)^{-3/2} \quad . \tag{3.1}$$

This corresponds to the result Hoel cites in his section 5.

This particular weight function is a bell-shaped curve, symmetric about the origin, and slightly more leptokurtic then the standard Normal p.d.f. It is a member of an entire family of densities, each of the form $f(x;m) = \Gamma(m)/[\Gamma(\tfrac{1}{2})\Gamma(m-\tfrac{1}{2})(x^2+1)^m]$ $(m \geq 1)$. This family is treated in some depth by Kendall and Stuart (1958, p. 59).

3.2. Segmented-Line Bands

As an alternative to the hyperbolic bands, Graybill and Bowden (1967) proposed bands made up of segments of straight lines. They also attain minimum width at the mean of the x_i's, and widen linearly as they go out. In our canonical setting, these bands follow the expression $\ell_{GB}(x) = 1 + |x|$, with $k = \Phi^{-1}[(1 + (1-\alpha)^{\frac{1}{2}})/2]$. As in section 3.1, we question what form of weight function produces this band.

Graybill and Bowden constructed their bands by starting with the probability statement $P[\hat\beta_0 - k \leq \beta_0 \leq \hat\beta_0 + k,\ \hat\beta_1 - k \leq \beta_1 \leq \hat\beta_1 + k] = 1 - \alpha$, and then noting that this produced simultaneous, rectangular confidence intervals for the regression parameters. In our notation this is $P[-k \leq w_j \leq k\ ,\ j = 0,1] = 1 - \alpha$. The corresponding solution set in w-space is, by construction, a square with sides of length 2k. See Figure 2.

In this case, $R_\ell(\theta)$ takes on a different expression depending on whether or not $\theta \leq \pi/4$. For $\theta \leq \pi/4$, the intersection of $w_1 = w_0 \tan\theta$ and $w_0 = k$ occurs for

$w_1 = k \tan\theta$. When $\theta > \pi/4$, intersecting $w_1 = w_0 \tan\theta$ and $w_1 = k$ gives $w_0 = k \cot\theta$. The results yield

$$R_\ell(\theta) = \begin{cases} k(1 + \tan^2\theta)^{\frac{1}{2}} & \text{if } \theta \le \pi/4 \\ k(1 + \cot^2\theta)^{\frac{1}{2}} & \text{if } \theta > \pi/4 \end{cases} .$$

This is a more complicated form for $R_\ell(\theta)$, and we need to be careful in using it to reexpress (2.3'):

$$\int_0^{\pi/4} \int_0^{k(1+\tan^2\theta)^{\frac{1}{2}}} re^{-r^2/2} dr d\theta + \int_{\pi/4}^{\pi/2} \int_0^{k(1+\cot^2\theta)^{\frac{1}{2}}} re^{-r^2/2} dr d\theta = \frac{1-\alpha}{2}\pi$$

results in ($x = \tan\theta$)

$$\int_0^1 \frac{\exp[-\frac{1}{2}k^2(1+x^2)]}{x^2+1} dx + \int_1^\infty \frac{\exp[-\frac{1}{2}k^2(x^2+1)/x^2]}{x^2+1} dx = \frac{\alpha\pi}{2} .$$

As in (2.11), when $\alpha \le 0.465068$, we can set

$$\tau_{GB}(x) = \begin{cases} \dfrac{\exp[-\frac{1}{2}k^2(x^2+1)]}{k(|x|+1)(x^2+1)} & \text{if } |x| \le 1 \\ \\ \dfrac{\exp[-\frac{1}{2}k^2(x^2+1)/x^2]}{k(|x|+1)(x^2+1)} & \text{if } |x| > 1 \end{cases} . \qquad (3.2)$$

For $\alpha = .05$, $k = 2.2365$. To satisfy (1.2) we use numerical quadrature to find $c = 28.4756$ (again, at $\alpha = .05$). A comparison of the weight functions (3.1) and (3.2) is given in Figure 3. Notice that $c\tau_{GB}$ is greater — i.e., gives greater weight — than $c\tau_{WH}$ around $x = 0$, drops below $c\tau_{WH}$ for x farther out, then crosses back above $c\tau_{WH}$ as $|x|$ grows large. As expected, the inverse of this occurs for the bands (see Graybill and Bowden, 1967, Figure 1, p. 407): $k\ell_{GB}$ is shorter than $k\ell_{WH}$ close to $\bar{x}$, widens past $k\ell_{WH}$ for intermediate x's, then crosses back inside of $k\ell_{WH}$ for extreme x.

4. MINIMAL $1-\alpha$ CONFIDENCE BANDS

One obvious setback with the exact approach outlined in section 2 is that once ℓ is specified, there can still be a great deal of computation involved in finding c_ℓ and $R_\ell(\theta)$. However, if we could find an upper bound on $R_\ell(\theta)$, for all ℓ, the resulting computations would simplify greatly. There is, of course, a trade-off here, since the bands would no longer have an exact confidence coefficient. Instead, they would have minimal confidence coefficient $\geq 1 - \alpha$.

4.1. Upper Bound for $R_\ell(\theta)$

Returning to the line intersections in Figure 1, we see that the maximum distance of any ray out from the origin is the distance to the point of intersection. From (2.4), this gives $R_\ell(\theta) \leq k\ell(\tan\theta)|\cos\theta|$. This changes (2.9) to

$$\int_0^\infty \frac{\exp[-\frac{1}{2}k^2\ell^2(x)/(x^2+1)]}{x^2+1}\,dx \leq \frac{\pi\alpha}{2} \ . \tag{4.1}$$

As in (2.10), we will minimize the difference between (4.1) and the integral of $c\tau(x)k\ell(x)$ over $(0,\infty)$. This difference is still bounded below by $z_{\alpha/2} - \frac{1}{2}\pi\alpha$, so that for any $\alpha \geq 0.465068$, we can take

$$c\tau(x) \doteq \frac{\exp[-\frac{1}{2}k^2\ell^2(x)/(x^2+1)]}{k\ell(x)(x^2+1)} \ , \tag{4.2}$$

subject to $\tau(x) = \tau(-x)$ and $\ell(x) = \ell(-x)$. As before, this gives an inverse relationship between τ and ℓ. Notice, also, that when $\ell(x)$ is the hyperbolic form $\ell_{WH}(x) = (1+x^2)^{\frac{1}{2}}$, (4.2) and (2.11) are identical.

4.2. Comparison of Minimal and Exact Formulations

The obvious question of interest is, how do (4.2) and (2.11) compare, i.e., how much do we lose when sacrificing an exact confidence coefficient for computational ease? As noted above, the formulations are equivalent for the Working-Hotelling bands, so the comparison there is trivial. Another possible comparison is available with the Graybill-Bowden bands of Section 3.2; $\ell_{GB}(x) = 1 + |x|$. Substituting this into (4.2) and simplifying gives

$$\tau_{GB}(x) \doteq \frac{\exp[-k^2|x|/(x^2+1)}{(x^2+1)(|x|+1)} \quad . \tag{4.3}$$

This form is enough to give us an impression of what weight function makes the (minimal) segmented-line bands τ-optimal. In order to make a better comparison between the minimal and exact formulations, we can normalize this expression using (1.2). Numerical quadrature, again at $\alpha = .05$, gives $c = 2.2856$.

As can be seen by comparing (4.3), properly normalized, with (3.2), there are values of x where the two formulations give drastically different values. Of course, there are some values for x where the two are rather close. (Some comparison values are given in Table 1. The two functions are presented in Figure 4.) Indeed, the minimal approach seems to provide an adequate approximation to the exact prior weight function over many values of $x \in \mathbf{R}$. Still, there is enough of a discrepancy evidenced between the two to suggest that the sacrifices involved in attaining more computational simplicity with the minimal approach can become rather severe.

5. DISCUSSION

Equations (2.11) and (4.2) illustrate an inverse relationship between a weight function, τ, and the band form, ℓ, that is τ-optimal against it. This concept is both intuitively appealing and easily conceptualized. For example, as τ's tails fatten, ℓ becomes flatter.

Experimenters can utilize these results to report confidence bands which reflect their prior interest. As seen in Section 3, any given band form can be specified for $\ell(x)$, and the corresponding constant k can be determined for a given α. It can then be determined, either exactly with (2.11) or approximately with (4.2), which prior weight function makes this particular form τ-optimal. In practice, this approach can be useful. An experimenter may have a good idea about the kind of band form he'd like to report. However, he may have little insight into choosing a proper mathematical form for τ. Given ℓ, the weight function against which it is τ-optimal can be determined. Then, the sensibility of such a choice can be evaluated.

REFERENCES

Bohrer, R. (1973). An optimality property of Scheffé bands. Ann. Statist. **1**, 766-772.

Bowden, D. C. (1970). Simultaneous confidence bands for linear regression models. J. Amer. Statist. Assoc. **65**, 413-421.

Casella, G. and Strawderman, W. E. (1980). Confidence bands for linear regression with restricted predictor variables. J. Amer. Statist. Assoc. **75**, 862-868.

Cima, J. A. and Hochberg, Y. (1976). On optimality criteria in simultaneous interval estimation. Commun. Statist. – Theory Meth. **A5**, 875-882.

Feller, W. (1968). An Introduction to Probability Theory and its Applications (v.1, 3rd ed.), New York: Wiley.

Gafarian, A. V. (1964). Confidence bands in straight line regression. J. Amer. Statist. Assoc. **59**, 182-213.

Graybill, F. A. and Bowden, D. C. (1967). Linear segment confidence bands for simple linear models. J. Amer. Statist. Assoc. 62, 403-408.

Hoel, P. G. (1951). Confidence regions for a linear relation. Proc. 2nd Berkeley Symp. on Math. Statist. and Prob., Berkeley: Univ. Calif. Press, 75-81.

Hunter, J. S. (1981). Calibration and the straight line: current statistical practices. J. Assoc. Off. Anal. Chem. 64, 574-583.

Kendall, M. G. and Stuart, A. (1958). The Advanced Theory of Statistics (v.1), London: C. Griffin and Co.

Khorasani, F. and Milliken, G. A. (1979). On the exactness of confidence bands about a linear model. J. Amer. Statist. Assoc. 74, 446-448.

Miller, R. G. (1981). Simultaneous Statistical Inference. (2nd ed.), New York: Springer-Verlag.

Naiman, D. Q. (1983a). Comparing Scheffé-type to fixed width confidence bands in regression. J. Amer. Statist. Assoc., to appear.

Naiman, D. Q. (1983b). μ-Optimality of simultaneous confidence bands (abstract). Inst. Math. Statist. Bull. 12, 75.

Trout, J. R. and Swallow, W. H. (1979). Regular and inverse interval estimation of individual observations using uniform confidence bands. Technometrics 21, 567-574.

Uusipaikka, E. (1983). Exact confidence bands for the regression line in simple linear regression. J. Amer. Statist. Assoc., to appear.

Wynn, H. P. and Bloomfield, P. (1971). Simultaneous confidence bands in regression analysis. J. Roy. Statist. Soc., Ser. B, 33, 202-217.

Working, H. and Hotelling, H. (1929). Applications of the theory of error to the interpretation of trends. J. Amer. Statist. Assoc. Suppl. 24, 73-85.

Zvára, K. (1979). On exact confidence regions for linear regression functions. Math. Oper. Statist., Ser. Statist. 10, 55-62.

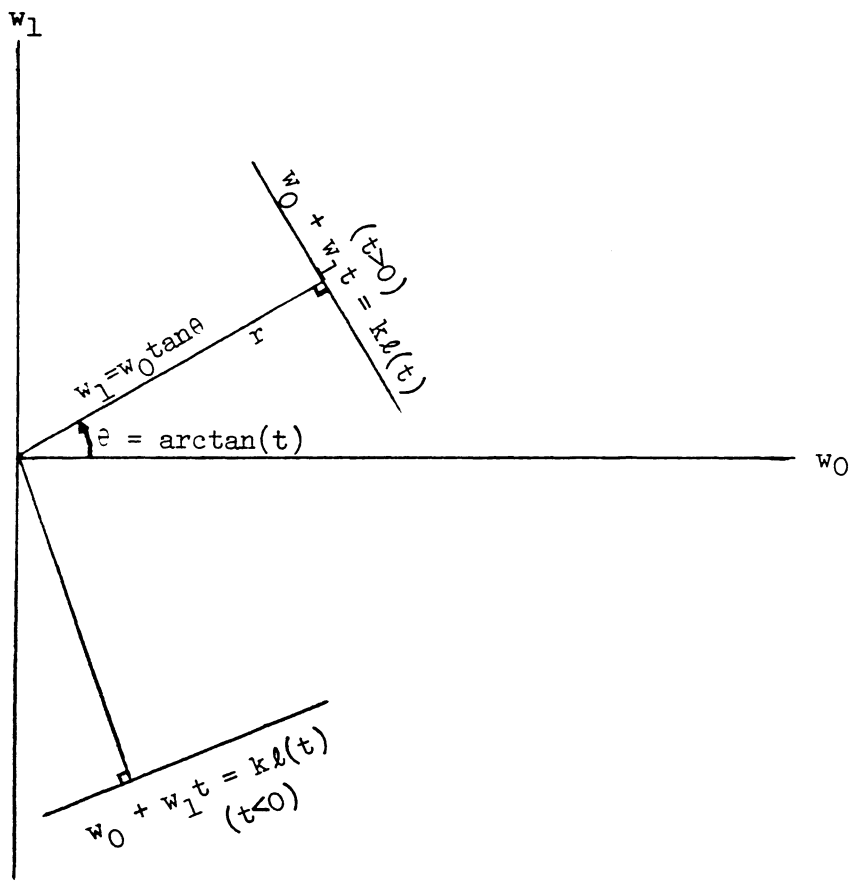

FIGURE 1: Line intersections in w-space.

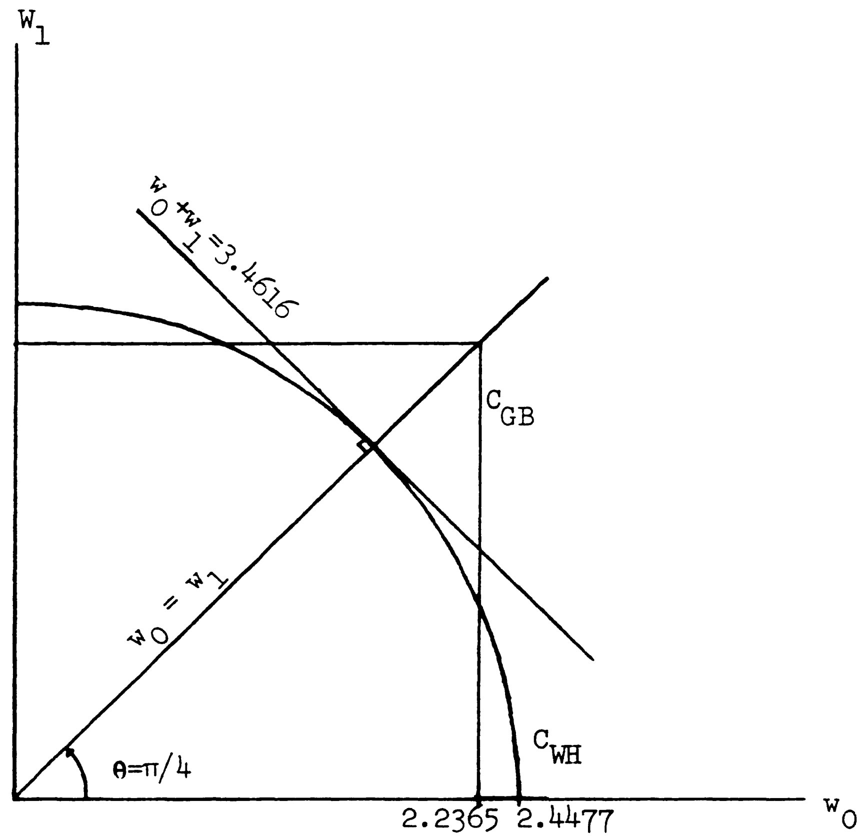

FIGURE 2: Solution sets in w-space for the Working-Hotelling, C_{WH}, and Graybill-Bowden, C_{GB}, bands (I^{st} quadrant only); $\alpha = .05$.

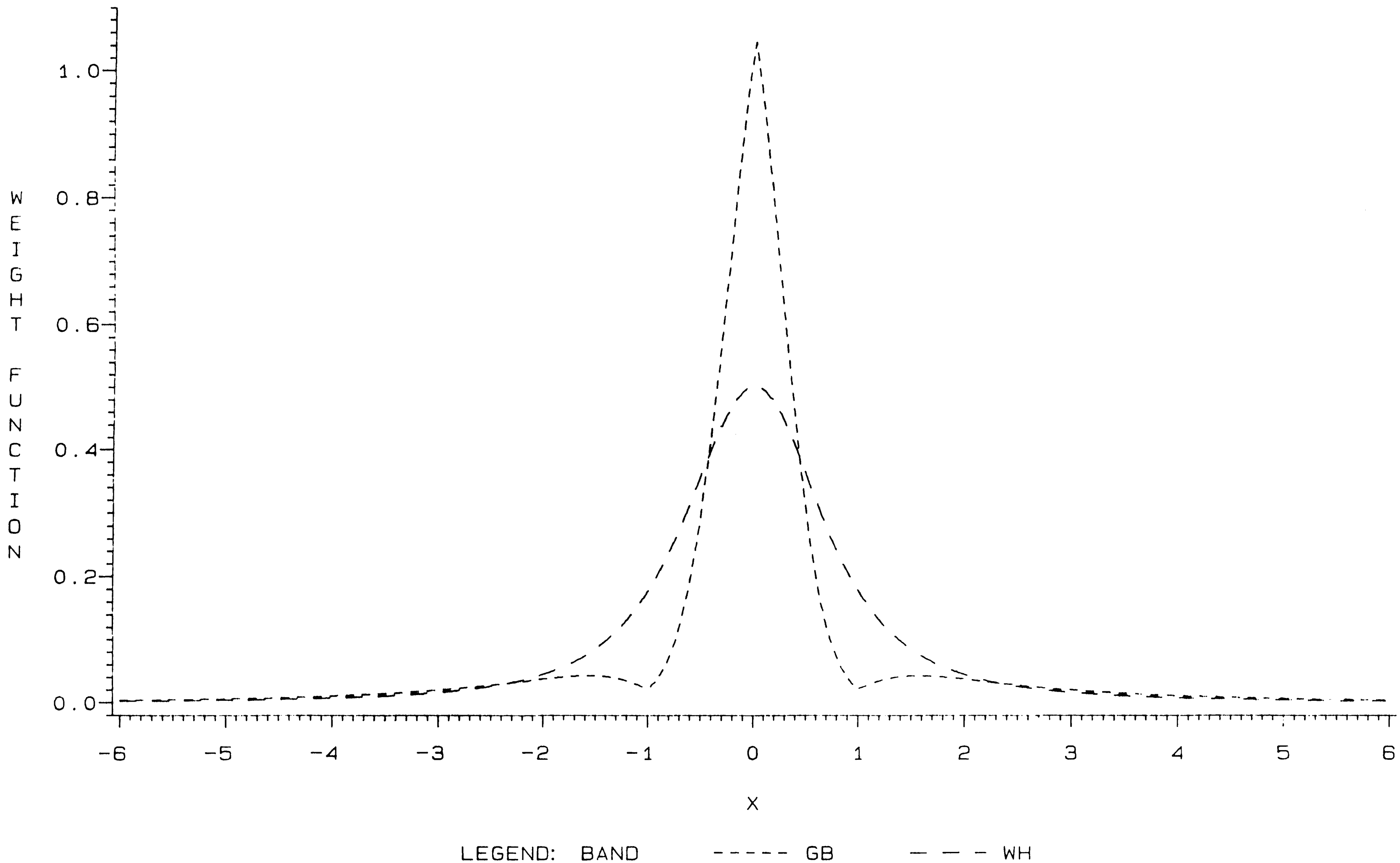

Figure 3. Weight functions for the Working–Hotelling (WH) and Graybill–Bowden (GB) confidence bands, $\alpha = .05$

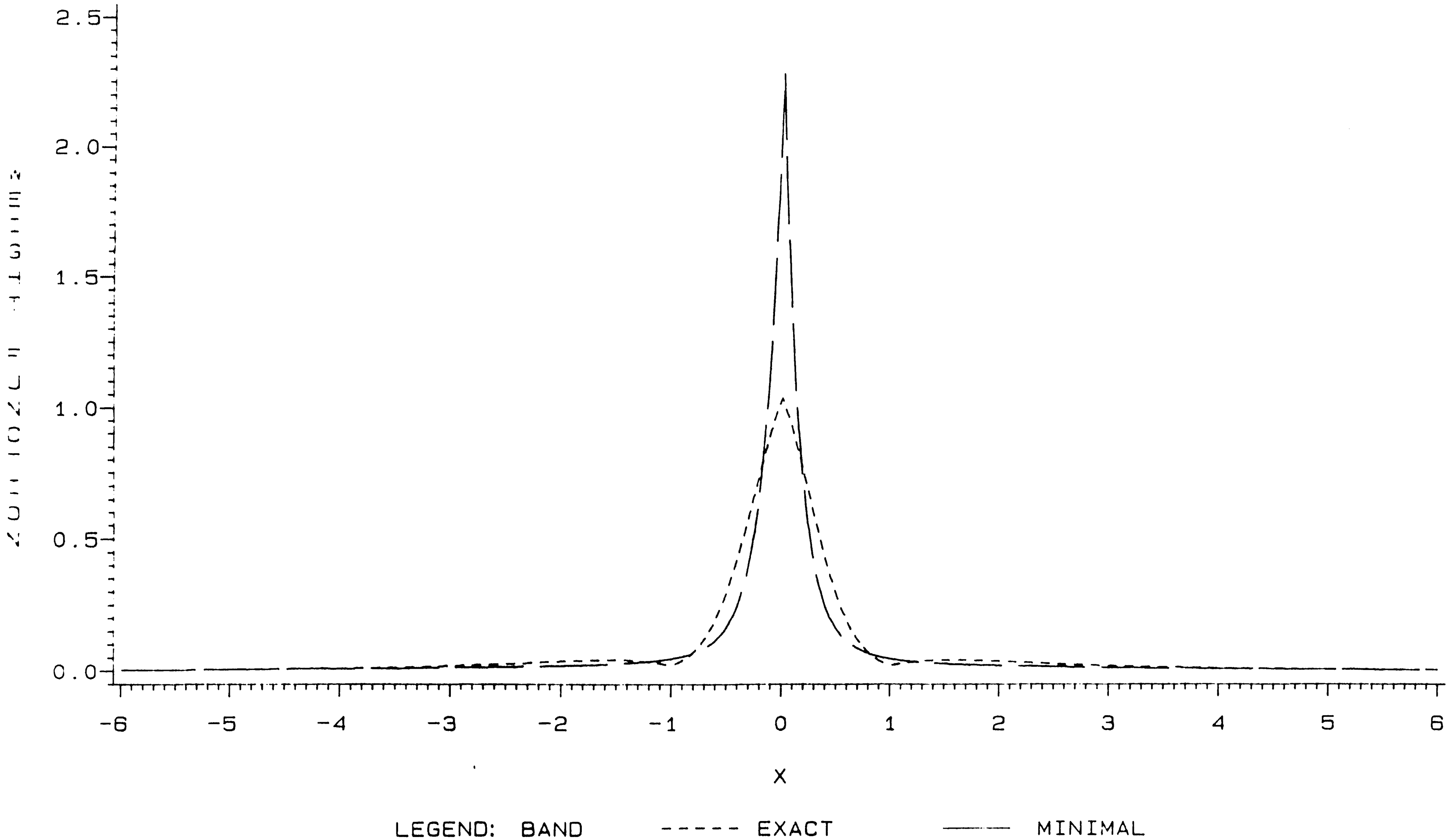

Figure 4. Weight functions for the Graybill–Bowden confidence bands, α = .05

Table 1. Comparison values between exact and minimal formulations for $c\tau_{GB}(x)$.

x	Exact	Minimal
0.0	1.044	2.286
0.2	0.757	0.700
0.4	0.431	0.251
0.6	0.195	0.116
0.8	0.071	0.067
1.0	0.021	0.047
1.2	0.034	0.036
2.0	0.037	0.021
3.0	0.020	0.013